

# Asymptotic behaviour of the $S$ -stopped branching processes with countable state space

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**Abstract:** The starting process with countable number of types  $\mu(t)$  generates a stopped branching process  $\xi(t)$ . The starting process stops, by falling into the nonempty set  $S$ . It is assumed, that the starting process is subcritical, indecomposable and noncyclic. It is proved, that the extinction probability converges to the cyclic function with period 1.

**Keywords:** branching processes; Markov chain; extinction probability; asymptotic behavior.

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**1.** Let us consider a measured state space  $(X, \mathcal{A})$ , where  $\mathcal{A}$  is the  $\sigma$ -algebra on  $X$ . On this space we consider unbreakable homogenous Markov process with transition probability  $P(t, x, A)$ , where  $t$  denotes time,  $x \in X$  and  $A \in \mathcal{A}$ . Considering every trajectory of the given process as an evolution of the motion of a particle,  $P(t, x, A)$  can be interpreted as a probability that a particle, which starts its motion from  $x \in X$ , falls into the set  $A \in \mathcal{A}$  till the time  $t$ . It is assumed, that the time is discrete and the lifetime of a particle is equal to 1. At the end of its life the particle promptly gives rise to a number of offsprings, starting position of which are randomly distributed on the space  $X$ . The number and the position of these offsprings depends only on

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the position of the particle-ancestor at the transformation time point. Further every offspring evolves analogously and independently of other particles.

Let  $\mu_{xt}(A)$  be a random measure, which for every  $A \in \mathcal{A}$  is equal to the number of the particles at time point  $t$ , types of which fall into set  $A$ , under condition that the process started with one particle  $x \in X$ .  $\mu_t(A)$  is a random measure equal to the number of particles at the time  $t$ , which types are from the set  $A$ , but without any knowledge about starting group of particles.

Further we assume, that the space  $X$  consists of a countable number of elements  $x_1, \dots, x_n, \dots$ . This means that the set of types of particles  $\{T_1, \dots, T_n, \dots\}$  is countable.

Based on the measure  $\mu_{xt}(A)$  we introduce the multivariate measure  $\boldsymbol{\mu}_{\mathbf{x}t}(A)$

$$\boldsymbol{\mu}_{\mathbf{x}t}(A) = \left\{ \begin{array}{ll} \sum_{i=0}^{\infty} \sum_{j=1}^{n_i} \mu_{x_{ij}t}(x_m), & \text{if } x_m \in A \\ 0, & \text{else} \end{array} \right\}_{m=0}^{\infty},$$

where  $\mathbf{x} = \{x_{11}, \dots, x_{1n_1}, x_{21}, \dots, x_{2n_2}, \dots\}$ ,  $x_{ij} \in X$  is the  $j$ -th element  $i$ -th type.

Let us denote  $\mathcal{N}_0 = \{0, 1, 2, \dots\}$ , and respectively  $\mathcal{N}_0^\infty$  is an infinite dimensional measured space with elements  $x_i \in \mathcal{N}_0$ .

Having  $P(t, x, A)$  let us introduce  $\hat{P}(t, \mathbf{x}, \mathbf{y})$ ,  $(\mathbf{x}, \mathbf{y} \in \mathcal{N}_0^\infty)$ , where  $\hat{P}$  is a probability that we obtain vector  $\mathbf{y}$  till time  $t$ , assuming that we started from  $\mathbf{x}$ .  $\hat{P}$  could be rewritten in terms of  $\boldsymbol{\mu}_{\mathbf{x}t}$  as

$$\hat{P}(t, \mathbf{x}, \mathbf{y}) = P\{\boldsymbol{\mu}_{\mathbf{x}t}(X) = \mathbf{y}\}.$$

Let  $\mathcal{E}(i) = (\delta_{i1}, \dots, \delta_{in}, \dots)$ , where  $\delta_{ij}$  is the Kronecker symbol,  $\delta_{\mathbf{x}\mathbf{y}} = \prod_{i=1}^{\infty} \delta_{x_i y_i}$ ,  $\mathcal{E}(i)$  is the particle of the  $i$ -th type. We also assume, that  $a^b = a_1^{b_1} a_2^{b_2} \dots a_n^{b_n} \dots$ ,  $a! = a_1! a_2! \dots a_n! \dots$ ,  $\bar{a} = a_1 + \dots + a_n + \dots$ ,  $a_i^{[b_i]} = a_i(a_i - 1) \dots (a_i - b_i + 1)$ .

**Definition 1.** *Functional*

$$F(s(\cdot)) = F(s) = \mathbb{E} \exp \left\{ \int \ln s(x) \mu(dx) \right\}$$

is called a generated functional of the random measure  $\mu$ , where  $s(x)$  is a measured bounded function.

Generated functional  $F(s)$  is always defined, when  $0 < |s(x)| \leq 1$  and integral  $\int \ln s(x) \mu(dx)$  exists.

For our process the generated functional is given as

$$h(t, s(\cdot)) = \mathbb{E} \exp \left\{ \int_X \ln s(z) \boldsymbol{\mu}_t(dz) \right\},$$

where  $\mu_t$  is the same multivariate measure as  $\mu_{\mathbf{x}t}$  but not taking into account any combination of the starting position of the process. Further we will consider the case  $s(\cdot) = \text{const} = s = (s_1, s_2, \dots)$ . It is easy to check whether the introduced generated functional is generated, as in the case of finite number of types (in that case it is not a functional but a function).

Let us denote

$$\begin{aligned} h^i(t, s) &= h^{\mathcal{E}(i)}(t, s), \\ h^{\beta}(t, s) &= ((h^{\mathcal{E}(1)}(t, s))^{\beta_1}, (h^{\mathcal{E}(2)}(t, s))^{\beta_2}, \dots), \\ h(t, s) &= (h^{\mathcal{E}(1)}(t, s), h^{\mathcal{E}(2)}(t, s), \dots). \end{aligned}$$

It is proved in [3], that the introduced generated function follows the main functional equation ( $\forall t, \tau = 0, 1, 2, \dots$ )

$$h(t + \tau, s) = h(t, h(\tau, s)).$$

Let us fix the finite subset  $S \subset \mathcal{N}_0^\infty$ ,  $0 \notin S$ . *Stopped* or *S-stopped* multitype branching process is the process  $\xi_{\mathbf{x}t}(X)$ , defined for  $t = 1, 2, \dots$  and  $\mathbf{x} \in \mathcal{N}_0^\infty$  by equations

$$\xi_{\mathbf{x}t}(X) = \begin{cases} \mu_{\mathbf{x}t}(X), & \text{if } \forall v, 0 \leq v < t, \mu_{\mathbf{x}v}(X) \notin S \\ \mu_{\mathbf{x}u}(X), & \text{if } \forall v, 0 \leq v < u, \mu_{\mathbf{x}v}(X) \notin S, \mu_{\mathbf{x}u}(X) \in S, u < t. \end{cases}$$

From this, for the *S-stopped* process  $\xi_{\mathbf{x}t}(X)$ , points of the set  $S$  are additional states of absorption compared to the process  $\mu_{\mathbf{x}t}(X)$ . The latter had only one point of absorption 0. In contrast to the process  $\mu_{\mathbf{x}t}(X)$ , in the *S-stopped* branching process  $\xi_{\mathbf{x}t}(X)$  single particles in generation  $t$  multiplies independently following probability law defined by the generated functional  $h(\cdot)$ , only if  $\xi_{\mathbf{x}t}(X) \notin S$ . If the random vector  $\xi_{\mathbf{x}t}(X)$  falls into the set  $S$ , the evolution of the process stops.

Since the process  $\mu_{\mathbf{x}t}(X)$  is a Markov chain, then

$$\hat{P}(t_1 + t_2, \alpha, \beta) = \sum_{\gamma \in \mathcal{N}_0^\infty} \hat{P}(t_1, \alpha, \gamma) \hat{P}(t_2, \gamma, \beta).$$

For further needs, we also consider probabilities  $\tilde{P}(t, \alpha, \mathbf{r})$ , defined as

$$\tilde{P}(t, \alpha, \mathbf{r}) = \begin{cases} \hat{P}(1, \alpha, \mathbf{r}), & t = 1; \\ \sum_{\beta \notin S} \hat{P}(1, \alpha, \beta) \tilde{P}(t-1, \beta, \mathbf{r}), & t \geq 2. \end{cases} \quad (1)$$

It is easy to see, that  $\tilde{P}(l, \alpha, \mathbf{r})$  is a conditional probability of the event

$$\{\mu_{\alpha l}(X) = \mathbf{r}\} \cap \left( \bigcap_{l'=1}^{l-1} \{\mu_{\alpha l'}(X) \notin S\} \right).$$

Let

$$q_{\mathbf{r}}^{\mathbf{n}}(t) = P\{\xi_{\mathbf{n}t}(X) = \mathbf{r}\}$$

be the probability of an extinction of the  $S$ -stopped branching process  $\xi_{\mathbf{x}t}(X)$  into state  $\mathbf{r} \in S$  till time  $t$ , starting from state  $\mathbf{n} \in \mathcal{N}_0^\infty$ .

## 2. Main facts.

**Theorem 1.** *For any  $\mathbf{n} \notin S$ ,  $\mathbf{n} \neq 0$ ,  $\mathbf{r} \in S$ ,  $t \geq 1$  holds*

$$q_{\mathbf{r}}^{\mathbf{n}}(t) = \sum_{\alpha \in S} \sum_{l=1}^t c_{\alpha\mathbf{r}}(t, l) \hat{P}(l, \mathbf{n}, \alpha), \quad (2)$$

where coefficients  $c_{\alpha\mathbf{r}}(t, l)$  can be found from

$$c_{\alpha\mathbf{r}}(t+1, l+1) = c_{\alpha\mathbf{r}}(t, l), \quad (3)$$

$$c_{\alpha\mathbf{r}}(t+1, 1) = \delta_{\alpha\mathbf{r}} - \sum_{l=1}^{t-1} \tilde{P}(l, \alpha, \mathbf{r}), \quad (4)$$

$$c_{\alpha\mathbf{r}}(1, 1) = \delta_{\alpha\mathbf{r}}. \quad (5)$$

*Proof.* Let

$$\tau = \min \{t : \mu_{\mathbf{n}t}(X) \in S\}$$

be the moment of the first fall into  $S$ , then for  $t \geq l$

$$P\{\xi_{\mathbf{n}t}(X) = \mathbf{r}, \tau = l\} = P\{\xi_{\mathbf{n}l}(X) = \mathbf{r}\} = \tilde{P}(l, \mathbf{n}, \mathbf{r}).$$

Applying (1) to  $\tilde{P}(l, \mathbf{n}, \mathbf{r})$ ,  $l \geq 2$ , we get

$$\begin{aligned} \tilde{P}(l, \mathbf{n}, \mathbf{r}) &= \sum_{\alpha \notin S} \hat{P}(1, \mathbf{n}, \alpha) \tilde{P}(l-1, \alpha, \mathbf{r}) \\ &= \sum_{\alpha \notin S} \hat{P}(2, \mathbf{n}, \alpha) \tilde{P}(l-2, \alpha, \mathbf{r}) - \sum_{\alpha \in S} \hat{P}(1, \mathbf{n}, \alpha) \tilde{P}(l-1, \alpha, \mathbf{r}). \end{aligned}$$

The first sum on the right hand side of this formula can be transformed similarly

$$\sum_{\alpha \notin S} \hat{P}(2, \mathbf{n}, \alpha) \tilde{P}(l-2, \alpha, \mathbf{r}) = \sum_{\alpha \notin S} \hat{P}(3, \mathbf{n}, \alpha) \tilde{P}(l-3, \alpha, \mathbf{r}) - \sum_{\alpha \in S} \hat{P}(2, \mathbf{n}, \alpha) \tilde{P}(l-2, \alpha, \mathbf{r}).$$

Making the same transformations in the sum  $\sum_{\alpha \notin S} \hat{P}(i, \mathbf{n}, \alpha) \tilde{P}(l-i, \alpha, \mathbf{r})$ , we get

$$\tilde{P}(l, \mathbf{n}, \mathbf{r}) = \hat{P}(l, \mathbf{n}, \mathbf{r}) - \sum_{\alpha \in S} \sum_{i=1}^{l-1} \hat{P}(l-i, \mathbf{n}, \alpha) \tilde{P}(i, \alpha, \mathbf{r}), \quad (6)$$

$$l = 2, \dots, t, \quad \tilde{P}(l, \mathbf{n}, \mathbf{r}) = \hat{P}(1, \mathbf{n}, \mathbf{r}). \quad (7)$$

As  $q_{\mathbf{r}}^{\mathbf{n}}(t) = \sum_{l=1}^t \tilde{P}(l, \mathbf{n}, \mathbf{r})$ , from formulas (6),(7) we get (3),(4),(5).  $\square$

Late on we will consider the process similarly to [1]. Let

$$A_1(x, D) = E\{\xi_{x1}(D)\}$$

be the first factorial moment, where  $\xi_{x1}(D)$  is such a random measure, which for each  $D \in \mathcal{A}$  is equal to the number of particles at time point 1, which types belong to set  $D$ , conditional on  $S$ -stopped process. It also taken into account that at the beginning there was only one particle of the type  $x \in X$ , what means  $\xi_{x1}(D) = \sum_{i=1}^{\infty} \xi_{x1i}(D)$ . From the linearity of  $E$  we have  $A_1(\mathbf{x}, D) = E\{\xi_{\mathbf{x}1}(D)\} = \sum_{i=1}^{\infty} A_1(x_i, D)$ . It is important that  $D$  could be a vector or a set.

**Definition 2.** Let  $A_1(x, D) = A(x, D)$  and

$$A_{n+1}(x, D) = \int_X A_n(y, D) dA(x, y) = \int_X A(y, D) dA_n(x, y).$$

It is assumed, that  $A_0(x, D) = 1$ , if  $x \in D$  and  $A_0(x, D) = 0$  else.

In [4] it is proved, that iterations of the operator  $A$  coincide with the first moments of  $\xi$ . This means, that for matrix of the linear operator  $A(t)$ , with  $A_{ij}(t) = A_t(x_i, x_j)$ , it holds that  $A(t) = A^t$  will take place, where  $A = A(1)$ .

Let

$$B_t(x, D_1, D_2) = E\{\xi_{xt}(D_1) \cdot \xi_{xt}(D_2) - \xi_{xt}(D_1 \cap D_2)\}$$

be the second factorial moment.

For further work we have to introduce some definitions, describing classes of branching processes (see [3]).

**Definition 3.** Branching process in which all types form a single class of equivalent types is called indecomposable. All other processes are called decomposable. Branching process is called fully indecomposable if the set of types could be split-up into two nonempty closed sets.

**Definition 4.** An indecomposable discrete time branching process is called cyclic with period  $d$ , if the greatest divisor for all  $t$ , such that  $\langle A_t(x_i, x_i) \rangle > 0$ , is equal to  $d$ . If  $d = 1$  then the process is called noncyclic.

**Definition 5.** An indecomposable discrete time branching process is called subcritical, if the largest eigenvalue (Perron's root)  $\delta$  of the matrix  $A$  is smaller than 1, supercritical, if  $\delta > 1$  and critical if  $\delta = 1$  and  $f(x_i)B_{jk}^i\nu(x_j)\nu(x_k) > 0$ , where  $B_{jk}^i$  is the matrix of the operator  $B$ , and  $f$  and  $\nu$  eigenfunction and invariant measure respectively which correspond do the Perron's root  $\delta$ .

**Assumption 1.** The kernel  $E\xi_{\mathbf{x}t}(S)$  is assumed to be indecomposable, noncyclic and subcritical.

Correspondingly to the assumption 1 the operator  $A$ , which is defined by the kernel  $E\{\xi_{xt}(D)\}$  in the space of measurable functions and in the space of measures, has the eigenfunction  $f(\cdot)$  and the invariant measure  $\nu(\cdot)$ , such that

$$\begin{aligned}\int_X f(y)A_t(x, dy) &= f(x) = \sum_{i=1}^{\infty} f(y_i)A_t(x, y_i), \\ \int_X A_t(x, Y)\nu(dx) &= \nu(Y) = \sum_{i=1}^{\infty} A_t(x_i, Y)\nu(x_i).\end{aligned}$$

Further we assume, that  $0 < x_1 < f(x) < x_2 < \infty$ ,  $\nu(X) < \infty$  and

$$\int_X f(y)\nu(dy) = 1 = \sum_{i=1}^{\infty} f(y_i)\nu(y_i). \quad (8)$$

The operator induced by the above defined kernel in the space of bounded functions has  $\{1\}$  as an isolated point of the spectrum.

**Assumption 2.** We assume  $E\{\mu_{\mathcal{E}(j)1}(x_i) \log \mu_{\mathcal{E}(j)1}(x_i)\}$  is finite for  $\forall i, j = 1, 2, \dots$

**Assumption 3.** The expansion  $A_t(\mathbf{x}, \mathbf{y}) = \sum_k f(x_k)\delta_k^t \nu(y_k)$  exists.

As in indecomposable, noncyclic, subcritical processes with discrete time all absolute values of eigenvalues are less than one, then based on the assumption 3 we can conclude, that when  $t \rightarrow \infty$

$$A_t(x_i, y_j) = f(x_i)\delta^t \nu(y_j) + o(\delta_1^t),$$

where  $\delta$  is the largest eigenvalue. Thus

$$\lim_{t \rightarrow \infty} A_t(x_i, y_j)\delta^{-t} = f(x_i)\nu(y_j). \quad (9)$$

Let us denote

$$\begin{aligned}R^i(t, s) &= 1 - h^i(t, s), \\ R(t, s) &= (R^1(t, s), \dots, R^n(t, s), \dots), \\ R(t, 0) &= Q(t) = (Q^1(t), \dots, Q^n(t), \dots) = \lim_{s \rightarrow 0} R(t, s).\end{aligned}$$

As in the case with the finite number of types, the following inequalities could be easily proved (see [3])

$$0 \leq R^i(t, s) \leq Q^i(t) \quad 0 < |s| \leq 1, \quad (10)$$

$$|R^i(t, s)| \leq 2Q^i(t) \quad 0 < |s| \leq 1. \quad (11)$$

(11) implies that for the degenerating branching processes  $R^i(t, s)$  converges uniformly to zero on  $0 < |s| \leq 1$ .

We need following technical assumption on the process

**Assumption 4.** Let  $A^t > 0$  for some  $t > 0$  in the sense  $\forall i, j \ a_{ij} > 0$  and  $h^i(t, s) \neq A_{ij}(t)$ .

Hereafter the notation  $A = \{a_{ij}\} > 0$ , means that  $a_{ij} > 0 \ \forall i, j$ , and the notation  $A > B$ , where  $A = \{a_{ij}\}, B = \{b_{ij}\}$  are matrices, means that  $a_{ij} > b_{ij} \ \forall i, j$ .

Let  $h(s) = h(1, s)$ .

**Assumption 5.** Following the above defined assumptions for this process, it holds that

$$1 - h(s) = [A - E(s)](1 - s), \quad (12)$$

where matrix  $E(s)$  with  $0 \leq s \leq s' \leq 1$  satisfies conditions  $0 \leq E(s') \leq E(s) \leq A$  and  $\lim_{s \rightarrow 1} E(s) = 0$ .

**Theorem 2.** With Assumptions 3-5

$$\lim_{t \rightarrow \infty} \frac{R^i(t, s)}{f(x_k)R^k(t, s)} = \nu(x_i)$$

uniformly on all  $s \neq 1, 0 \leq s \leq 1$ .

This theorem is proved analogically to theorem 1 on page 192 in [3], by replacing the right and left eigenvectors by eigenfunction and invariant measure respectively. Matrices are from the class of matrices of infinite measurable linear operator.

**Theorem 3.** By assumptions 1-5 for any  $i, j = 1, 2, \dots$  and for  $l \rightarrow \infty$  probability that the process extinct to 0 from one particle of type  $j$  over  $l$  is

$$1 - \hat{P}(l, \mathcal{E}(j), 0) = K(S_j)\delta^l(1 + o(1)), \quad K(S_j) > 0; \quad (13)$$

a) the limit of the conditional probabilities exists

$$\lim_{t \rightarrow \infty} P\{\mu_{nt}(X) = \mathbf{k} | \mathbf{n} \neq 0\} = p_{\mathbf{k}}^*, \quad (14)$$

and the generating function  $h^*(s) = \sum_{\mathbf{k} \in \mathcal{N}_0^\infty} p_{\mathbf{k}}^* s^{\mathbf{k}}$  is not depending on  $\mathbf{n}$  and satisfies the relationships

$$\begin{aligned} 1 - h^*(h(\cdot)) &= \delta(1 - h^*(s)), \\ h^*(0, \dots, 0, \dots) &= 0, h^*(1, \dots, 1, \dots) = 1; \end{aligned} \quad (15)$$

b) distribution  $p_{\mathbf{k}}^*$  has positive expectation

$$h_j^*(1) = \lim_{s \rightarrow 1} h_j^*(s) = \sum_{\mathbf{k} \in \mathcal{N}_0^\infty} k_j p_{\mathbf{k}}^*,$$

where  $h_j^*(s) = \frac{\partial h^*(s)}{\partial s_j}$ .

It is proved by mimicking the theorem 3 on page 198 from [3] with the use of theorem 2 for the representation of the limit of the generating function of the conditional distribution by getting result similar to one in [2].

Let us fix one more assumption

**Assumption 6.** Let  $h_{ij}(s) = \frac{\partial h_i(s)}{\partial s_j}$ , then for all  $j$ ,  $1 \leq j < \infty$  there exists such  $i$ ,  $1 \leq i < \infty$ , that  $h_{ij}(0)$  are positive.

From the equality

$$h_{ij}(0) = \hat{P}(0, \mathcal{E}(i), \mathcal{E}(j)) = P\{\mu_{\mathcal{E}(i)1}(X) = \mathcal{E}(j)\}$$

this means, that the corresponding probabilities  $\hat{P}(0, \mathcal{E}(i), \mathcal{E}(j))$  are positive.

To proceed further we need following lemma

**Lemma 1.** Under the assumptions 1-6, the limit of conditional probabilities is positive, for all  $i = 1, 2, \dots$

$$\lim_{t \rightarrow \infty} P\{\mu_{nt}(X) = \mathcal{E}(i) | n \neq 0\} = p_{\mathcal{E}(i)}^* > 0,$$

*Proof.* The generating function  $h^*(s) = \sum_k p_k^* s^k$  in Theorem 3 satisfies the equation (15). If we replace in this equation  $s$  by  $h(s)$ , and repeat this replacement  $t$  times, we get the equality

$$1 - h^*(h(t, s)) = \delta^t(1 - h^*(s)), \quad (16)$$

where  $h(t, s)$  is  $t$ -th iteration of the function implied by the main differential equation. By differentiating (16) with respect to  $s_j$  at  $s = 0$ , we obtain

$$\sum_{i=1}^{\infty} h_i^*(h(t, 0)) h_{ij}(t, 0) = \delta^t h_j^*(0) = \delta^t p_{e(j)}^*. \quad (17)$$

As all coordinates of  $h(t, 0)$  converge to 1, for  $t \rightarrow \infty$ , then by the theorem 3 we can find such  $T$  and  $C_1$ , that  $h_i^*(h(t, 0)) \geq C_1 > 0$  for  $t > T$ . According to the assumption 6 this implies that for all  $1 \leq j \leq \infty$  we can found such  $i$ , that  $h_{ij}(t, 0) > 0$ . For all  $i_1, i_2, \dots, i_{t+1}$  holds

$$h_{i_1 i_{t+1}}(t, 0) \geq \prod_{l=1}^t h_{i_l i_{l+1}}(0).$$

Thus (17) implies

$$\delta^t p_{e(j)}^* \geq C_1 \sum_{i=1}^t h_{ij}(t, 0) > 0, \forall 1 \leq j \leq \infty.$$

□



**Theorem 4.** By the assumption 1 the limiting extinction probabilities  $q_{\mathbf{r}}^{\mathbf{n}} = \lim_{t \rightarrow \infty} q_{\mathbf{r}}^{\mathbf{n}}(t)$ ,  $\forall \mathbf{n} \notin S$ ,  $\mathbf{r} \in S$ , can be written in the series representation

$$q_{\mathbf{r}}^{\mathbf{n}} = \sum_{l=1}^{\infty} \sum_{\boldsymbol{\alpha} \in S} c_{\boldsymbol{\alpha}\mathbf{r}} \hat{P}(l, \mathbf{n}, \boldsymbol{\alpha}), \quad (18)$$

where  $c_{\boldsymbol{\alpha}\mathbf{r}} = \lim_{t \rightarrow \infty} c_{\boldsymbol{\alpha}\mathbf{r}}(t, l) = \delta_{\boldsymbol{\alpha}\mathbf{r}} - \sum_{u=1}^{\infty} \tilde{P}(u, \boldsymbol{\alpha}\mathbf{r})$ .

*Proof.* Probabilities  $q_{\mathbf{r}}^{\mathbf{n}}(t)$  increase with  $t$  and are bounded above by 1. Then the limit  $q_{\mathbf{n}}^{\mathbf{r}} = \lim_{t \rightarrow \infty} q_{\mathbf{n}}^{\mathbf{r}}(t)$  exists.

We can pass to the limits on the left and on the right hand sides of the formula (2), when  $t \rightarrow \infty$ , as for all  $\boldsymbol{\alpha}, \mathbf{r} \in S$  holds that  $\tilde{P}(l, \boldsymbol{\alpha}, \mathbf{r}) \leq \hat{P}(l, \boldsymbol{\alpha}, \mathbf{r})$  and Chebyshev inequality and assumption 3 imply that

$$\begin{aligned} \hat{P}(l, \boldsymbol{\alpha}, \mathbf{r}) &\leq P \left\{ \sum_{j=1}^{\infty} \mu_{\boldsymbol{\alpha}l}(\mathcal{E}(j)) \geq 1 \right\} \\ &\leq \sum_{j=1}^{\infty} E \{ \mu_{\boldsymbol{\alpha}l}(\mathcal{E}(j)) \} \\ &= \sum_{i=1}^{\infty} \alpha_i \sum_{j=1}^{\infty} d_{ij} \delta^l (1 + o(1)). \end{aligned}$$

This means that series  $\sum_l \tilde{P}(l, \boldsymbol{\alpha}, \mathbf{r})$  and  $\sum_l \hat{P}(l, \boldsymbol{\alpha}, \mathbf{r})$  converge to each other. This implies (18).  $\square$

As in [2] let us consider the asymptotic behavior of  $q_{\mathbf{r}}^{\mathbf{n}}$  for  $\bar{\mathbf{n}} \rightarrow \infty$ .

**Theorem 5.** Let assumptions 1-3 are fulfilled and  $\lim_{\bar{\mathbf{n}} \rightarrow \infty} (n_i / \bar{\mathbf{n}}) = a_i$ , where  $a = (a_1, a_2, \dots)$ . In this case for  $\mathbf{r} \in S$  and  $\bar{\mathbf{n}} \rightarrow \infty$

$$q_{\mathbf{r}}^{\mathbf{n}} - H(\log_{\delta} \bar{\mathbf{n}}) \rightarrow 0, \quad (19)$$

where  $H(x)$  is a cyclic function with period 1, defined through the following equalities

$$\begin{aligned} H(x) &= \sum_{j=1}^{r_0} c_j H_j(x), \\ H_j(x) &= \sum_{L=-\infty}^{\infty} \delta^{j(L+x)} e^{-(\mathbf{a}, K) \delta^{L+x}}, \end{aligned}$$

where  $(\mathbf{a}, K) = \sum_{i=1}^{\infty} a_i K_i$ ,  $K_i$  as in (13),  $r_0 = \max\{\bar{\mathbf{r}} = r_1 + r_2 + \dots : \mathbf{r} \in S\}$ . Constants  $c_j = c_j(\mathbf{r}, \mathbf{a}, p^*)$  depend on  $\mathbf{r}$ ,  $\mathbf{a}$  and the limit distribution  $p^* = \{p_k^*\}$  which is defined in lemma 1.

*Proof.* Let  $\theta(l) = (\theta_1(l), \theta_2(l), \dots)$  be a random vector, which components  $\theta_i(l)$  are equal to the number of particles of type  $i$  which give an offspring to the  $l$ -th generation. Thus we can write, that for all  $\alpha \in S$ ,  $l \geq 1$   $\mathbf{n} \notin S$ , we have

$$\begin{aligned}\widehat{P}(l, \mathbf{n}, \alpha) &= \sum_{\{\beta: 1 \leq \bar{\beta} \leq \bar{\alpha}\}} P\{\mu_{\mathbf{n}, l}(X) = \alpha, \theta(0, l) = \beta\} \\ &= \sum_{\{\beta: 1 \leq \bar{\beta} \leq \bar{\alpha}\}} P\{\theta(0, l) = \beta\} P\{\mu_{\beta l}(X) = \alpha \mid \theta(0, l) = \beta\}.\end{aligned}\quad (20)$$

Under the assumptions of the theorem 5

$$\begin{aligned}P\{\theta(0, l) = \beta\} &= \prod_{i=1}^{\infty} \binom{n_i}{\beta_i} (\widehat{P}(l, \mathcal{E}(i), 0))^{n_i - \beta_i} (1 - \widehat{P}(l, \mathcal{E}(i), 0))^{\beta_i} \\ &= \bar{\mathbf{n}} \bar{\beta} \frac{\alpha \beta}{\beta!} K \beta \delta^l \bar{\beta} e^{-(\mathbf{a}, K) \bar{\mathbf{n}} \delta^l (1 + o(1))} (1 + o(1)),\end{aligned}\quad (21)$$

and the probability, not depending on  $\mathbf{n}$

$$\begin{aligned}&P\{\mu_{\beta l}(X) = \alpha \mid \theta(0, l) = \beta\} \\ &= \sum_{\{\alpha^{(jk)}\}} \prod_{k=1}^{\infty} \prod_{j=1}^{\beta_k} P\{\mu_{\mathcal{E}(k), l}^{jk}(X) = \alpha^{(jk)} \mid \mathcal{E}(k) \neq 0\} \\ &\longrightarrow \sum_{\{\alpha^{(jk)}\}} \prod_{k=1}^{\infty} \prod_{j=1}^{\beta_k} p_{\alpha^{(jk)}}^* \quad l \rightarrow \infty,\end{aligned}\quad (22)$$

where  $\mu_{\mathcal{E}(k), l}^{jk}(X)$  are branching processes, with the same distribution as  $\mu_{\mathcal{E}(k), l}(X)$ . The summation in  $\sum_{\{\alpha^{(jk)}\}}$  is done over all such  $\alpha^{(jk)}$ , which  $\sum_{k=1}^{\infty} \sum_{j=1}^{\beta_k} \alpha^{(jk)} = \alpha$ . The statements in (20)-(22) imply, that the general component of the series (18) for  $\bar{\mathbf{n}} \rightarrow \infty$ ,  $l \rightarrow \infty$  can be written in the form

$$\begin{aligned}(1 + o(1)) \sum_{\alpha \in S} \sum_{\{\beta: 1 \leq \bar{\beta} \leq \bar{\alpha}\}} g(\alpha, \beta) \sum_{\bar{\beta}}^{r_0} \delta^{(l + \log_{\delta} \bar{\mathbf{n}})} \bar{\beta} \\ \times \exp \left\{ -(\mathbf{a}, K) \delta^{l + \log_{\delta} \bar{\mathbf{n}}} (1 + o(1)) \right\},\end{aligned}\quad (23)$$

where  $g(\alpha, \beta)$  is an independent of  $\mathbf{n}$  and  $l$  function. It is easy to see that in formula (18) for  $\bar{\mathbf{n}} \rightarrow \infty$  each component of series with any  $l \geq 1$  converges to zero.

Let us choose  $L_1 < L_2$  in such way that sums

$$\sum_{L=L_2}^{\infty} \delta \bar{\beta}^L e^{-(\mathbf{a}, K) \delta^L} \text{ and } \sum_{L=-\infty}^{L_2} \delta \bar{\beta}^L e^{-(\mathbf{a}, K) \delta^L} \quad (24)$$

are small. We set  $l_i + \log_{\delta} \bar{\mathbf{n}} = L_i + x_{i\bar{\mathbf{n}}}$ , for  $i = 1, 2$ , where  $0 \leq x_{i\bar{\mathbf{n}}} \leq 1$ . (23) and (24) imply, that we can choose such  $L_1$ ,  $L_2$  and  $n_0$ , that tails of the sum in (18), bounded

from 1 to  $l_1$  and from  $l_2$  to infinity, are less than  $\varepsilon/2$ , where  $\varepsilon > 0$  is small. Elements of the series (18) with  $l_1 < l < l_2$  can be replaced by a limited expressions (23) for  $\bar{\mathbf{n}} \rightarrow \infty$  as well as for  $l \rightarrow \infty$ . The number of summands in the sum  $\sum_{l=l_1+1}^{l_2-1}$  in expression (18) is finite  $l_2 - l_1 - 1 = L_2 - L_1 - 1$ . This means that  $n_0$  can be chosen in such a way, that for all  $\mathbf{n} > n_0$  the approximation error will be also less than  $\varepsilon/2$ . This implies the statement of the theorem, while  $\varepsilon > 0$  is any real number.  $\square$

From the theorem it cannot be concluded directly, whether the coefficients  $c_j$  in the formula (19) are such, that  $H(x) > 0$ . For this we introduce the next lemma.

**Lemma 2.** *Under assumptions 1-6, there exists such a constant  $\Theta > 0$ , that for some number  $n_0$*

$$q_{\mathbf{r}}^{\mathbf{n}} > \Theta, \text{ for } \forall \mathbf{n} \text{ with } \bar{\mathbf{n}} \geq n_0 \text{ and } \forall \mathbf{r} \in S.$$

*Proof.* As for any  $t$ ,  $q_{\mathbf{r}}^{\mathbf{n}} = \lim_{t \rightarrow \infty} q_{\mathbf{r}}^{\mathbf{n}}(t) \geq q_{\mathbf{r}}^{\mathbf{n}}(t)$ , it is enough to prove, that the inequality  $q_{\mathbf{r}}^{\mathbf{n}}(t) \geq \Theta > 0$  holds for any large enough  $t$ , for all  $\mathbf{r} \in S$  and  $\mathbf{n}$  from  $\bar{\mathbf{n}} \geq n_0$ . Let us use the upper defined random vector  $\theta(0, t)$  and introduce one more random vector  $\theta'_i(t-1) = (\theta'_1(t-1), \theta'_2(t-1), \dots)$ , where  $\theta'_i(t-1)$  is the number of starting particles of  $i$ -th type, with nonempty offspring set in the  $(t-1)$ -th generation, but empty in the  $t$ -th generation. For  $\mathbf{n} = (n_1, n_2, \dots)$ ,  $n_1 \geq r_0 + 1$ , where  $r_0 = \max_{\mathbf{r} \in S} \bar{\mathbf{r}}$ , we use the inequality

$$\begin{aligned} q_{\mathbf{r}}^{\mathbf{n}}(t) &= P\{\xi_{\mathbf{n}t}(X) = \mathbf{r}\} \\ &\geq P\{\mu_{\mathbf{n}t}(X) = \mathbf{r}, \theta'(t-1) = (r_0 + 1 - \bar{\mathbf{r}}), \theta(0, t) = \bar{\mathbf{r}}\mathcal{E}(1)\}. \end{aligned} \quad (25)$$

The right side of (25) we write as a product of  $\mathcal{P}_1(\mathbf{n}, t)\mathcal{P}_2(t)$ , where  $\mathcal{P}_1(\mathbf{n}, t) = P\{\theta'(t-1) = (r_0 + 1 - \bar{\mathbf{r}})\mathcal{E}(1), \theta(0, t) = \bar{\mathbf{r}}\mathcal{E}(1)\}$  and depends on  $\mathbf{n}$  and  $t$ , but  $\mathcal{P}_2(t) = P\{\mu_{\mathbf{n}t}(X) = \mathbf{r} | \theta'(t-1) = (r_0 + 1 - \bar{\mathbf{r}})\mathcal{E}(1), \theta(0, t) = \bar{\mathbf{r}}\mathcal{E}(1)\}$  and depends only on  $t$ . From the definition of the random vectors  $\theta(0, t)$  and  $\theta'(t-1)$  we have, that

$$\begin{aligned} \mathcal{P}_1(\mathbf{n}, t) &= \frac{n_1!}{(n_1 - r_0 - 1)!(r_0 + 1 - \bar{\mathbf{r}})!\bar{\mathbf{r}}} [\hat{P}(t-1, \mathcal{E}(1), 0)]^{n_1 - r_0 - 1} \\ &\times (1 - \hat{P}(t-1, \mathcal{E}(1), 0))^{r_0 + 1 - \bar{\mathbf{r}}} (1 - \hat{P}(t-1, \mathcal{E}(1), 0))^{\bar{\mathbf{r}}} \\ &\times \prod_{i=1}^{\infty} [\hat{P}(1, \mathcal{E}(1), 0)^{n_i}] P\{\mu'_{r_0+1-\bar{\mathbf{r}}t}(X) = 0 | r_0 + 1 - \bar{\mathbf{r}} \neq 0\}; \end{aligned} \quad (26)$$

$$\mathcal{P}_2(t) = \prod_{k=1}^{\infty} \prod_{j=1}^{r_k} P\{\mu_{\mathcal{E}(1)t}^{(jk)}(X) = \mathcal{E}(k) | \mathcal{E}(k) \neq 0\}. \quad (27)$$

Here  $\mu, \mu', \mu^{(jk)}$  are branching processes, whose evolution is defined by a generating function  $h(s) = (h_1(s), h_2(s), \dots)$ . Setting  $t \rightarrow \infty$ , in such a way, that  $\bar{\mathbf{n}}\delta^t \rightarrow V > 0$  for  $\bar{\mathbf{n}} \rightarrow \infty$ , we get in the right side of the equality (26) a positive constant multiplied

by a conditional probability, which stays at the end of formula. Using the limiting relationship  $P\{\boldsymbol{\mu}'_t(X) = \mathbf{k} \mid \mathbf{k} \neq 0\} \rightarrow p_{\mathbf{k}}^*$ , of the theorem 3 and the equality

$$\sum_{\mathbf{k} \in \mathcal{N}_0^\infty} p_{\mathbf{k}}^* (\widehat{P}^{k_1}(1, \mathcal{E}(1), 0) \widehat{P}^{k_2}(1, \mathcal{E}(2), 0) \cdots) = h^*(h(0))$$

we have that this conditional probability is equal in limit to  $h^*(h(0))$ . Expression (27) does not depend on  $\mathbf{n}$  and is equal to the product  $\prod_{i=1}^\infty [p_{\mathcal{E}(i)}^*]^{r_i}$ , for  $t \rightarrow \infty$ . From lemma 1 this product is positive. That completes the proof.  $\square$

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